

FRACTIONAL BROWNIAN FLOWS*

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Abstract

We consider stochastic flow on \mathbb{R}^n driven by fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and study tangent flow and the growth of the Hausdorff measure of sub-manifolds of \mathbb{R}^n as they evolve under the flow.

The main result is a bound on the rate of (global) growth in terms of the (local) Hölder norm of the flow.

1 Introduction

Our main objective is to study the global geometric properties of a manifold embedded in Euclidean space, as it evolves under a stochastic flow of diffeomorphisms driven by a non diffusive process. This follows on from our previous paper [19] in which we obtained precise estimates for the rate of growth of the *Lipschitz-Killing curvatures*¹ of smooth, $(n - 1)$ -dimensional manifolds embedded in \mathbb{R}^n , as they evolve under an isotropic Brownian flow. In this paper, however, we turn to the non-Markovian, non-diffusive situation in which the flow is driven by fractional Brownian motions.

Although extensive literature is available for stochastic flows driven by standard Brownian motion (see [5, 13]), very little is known when the driver of the flow is changed to a non-Markovian, non-diffusive process, such as fractional Brownian motion. For instance, some of the very basic results concerning the tangent flow are yet to be unearthed in the case when the flow is driven by fractional Brownian motion. Here we intend to target precisely this aspect of the flow on our way to the main result of this paper.

Recall that a fractional Brownian motion $\{B^H(t), t \geq 0\}$ with Hurst parameter $H \in (0, 1)$, is the zero mean Gaussian process with covariance function

$$E[B^H(s)B^H(t)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (1)$$

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¹For a detailed exposition on Lipschitz-Killing curvatures, we refer the reader to [1].

When $H = 1/2$, B^H is the standard Brownian motion, which is a Markov process and also a martingale. However for $H \neq 1/2$, B^H is neither a Markov process, nor a semi-martingale.

In order to construct a non-diffusive flows, we start with a collection of independent fractional Brownian motions, $\{B_\gamma^H\}_{\gamma \in \mathbb{N}}$, a collection $\{U_\gamma\}_{\gamma \in \mathbb{N}}$ of deterministic vector fields on \mathbb{R}^n , and define, for some fixed but generic set $I \subset \mathbb{N}$ with $|I| < \infty$, where $|I|$ denotes the cardinality of I ,

$$U_I(x, t) = \sum_{\gamma \in I} U_\gamma(x) B_\gamma^H(t). \quad (2)$$

The flow of diffeomorphisms $\Phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$, or, equivalently, the stochastic flow driven by a fractional Brownian motion, can then be defined pointwise by setting

$$\Phi_t(x) = x + \sum_{\gamma \in I} \int_0^t U_\gamma(\Phi_s(x)) dB_\gamma^H(s). \quad (3)$$

Clearly, we shall need to place conditions on the vector fields for the result to give a diffeomorphism, but, prior to that, we need to make sense of the stochastic integrals here.

For $H = \frac{1}{2}$, the integral can be interpreted in either the Itô or a Stratonovich sense. When $H \neq \frac{1}{2}$ the standard semimartingale arguments cease to work and we have to make a choice of definition. There is a plethora of literature on various ways to define an integral $\int_a^b f(s) dB^H(s)$, where f is random and B^H the fractional Brownian motion. See, for instance, [2, 6, 8, 10, 14, 20].

We shall adopt the pathwise definition given by Zähle [20, 21], based on which Nualart and Răşcanu ([16]) proved existence and uniqueness of the solutions of multidimensional stochastic differential equations of the form

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB^H(s) + \int_0^t b(s, X_s) ds,$$

for $H > \frac{1}{2}$. Using this, Decreusefond and Nualart in [9] established the existence of a homeomorphic stochastic flow driven by fractional Brownian motion, and so our flows are well defined. We note here that stochastic integrals can also be defined for $H < 1/2$ using Malliavin calculus (see [8]), but existence of solution of stochastic integral equations of above type is not ensured.

Note that in (3) we do not have a drift part, as we intend to study flows driven purely by noise, which goes hand in hand with the way stochastic flows have been defined in [13].

Now that it is clear, in principle, which flows we are considering, we can turn to the geometry. Consider a fixed m -dimensional, ($m < n$) C^2 manifold embedded in \mathbb{R}^n , and consider its image under Φ , setting

$$M_t = \Phi_t(M) = \{x \in \mathbb{R}^n : x = \Phi_t(y) \text{ for some } y \in M\}.$$

Our interest is how M_t behaves as a function of t .

Although in [19] we were able to obtain information on all the Lipschitz-Killing curvatures of M_t , in the current, non-diffusion scenario everything is much harder, and so we shall suffice by studying only the size of M_t , as measured through its m -dimensional Hausdorff measure, $\mathcal{H}_m(M_t)$, which basically measures the m -dimensional Lebesgue measure of the set M_t . Our main result is Theorem 3.5, however one can see the main flavour of the result already for a flow driven by a single fractional Brownian motion. In this case we have

Theorem 1.1 *In the notation above, assuming that $|I| = 1$ in (3), and under conditions (A1)–(A3) of Section 2 on the vector field U , for every $\beta < H$ and $H > \frac{1}{2}$ there exist constants c_1 and C_1 , such that*

$$\sup_{t \in [0, T]} \mathcal{H}_m(M_t) \leq c_1 \mathcal{H}_m(M) 2^{C_1 T} \|B^H\|_{\beta, T}^{1/\beta},$$

where $\|B^H\|_{\beta, T}$ is the β -Hölder norm of B^H (cf. (10)).

It is not hard to see that the Hölder norm $\|B^H\|_{\beta, T}^{1/\beta}$ grows no faster than $O(T^{1+\epsilon})$ for any $\epsilon > 0$, so that the overall rate of growth of $\mathcal{H}_m(M_t)$ given by Theorem 1.1 is $O(2^{CT^{2+\epsilon}})$. One should hope for something that was smaller, and H -dependent, but current techniques fail to establish this.

Similarly, recent results of Baudoin and Coutin [4] seem to indicate that correct growth rate should be $O(2^{CT^{2H}})$. These results, however, are based on the rough path approach of [3, 15]. While Hairer and Ohashi in [12] have proved existence of a stationary solution of (3), under conditions on the vector fields U_γ and assuming that for $|I| < \infty$, an approach via rough paths also seems unable to reach a better growth rate.

Our proof of Theorem 1.1 and the more general Theorem 3.5 will be based on the approach of Hu and Nualart [17], who obtained growth estimates on the solution of (3). The details follow in the remaining two sections.

In Section 2, apart from being more formal about setting up notation, we shall recall some basic formulae from the fractional calculus required for our main analysis. Estimates on the tangent flow and the flow itself, together with the proof of the main results, will form the bulk of Section 3.

2 Preliminaries

We start by listing some of the basic formulae required from the deterministic fractional calculus, and the fractional spaces associated with them. (See [20, 21] for a complete account of fractional calculus.)

For $a, b \in \mathbb{R}$, $a < b$, let $L^p(a, b)$, $p \geq 1$, be the space of Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$ with $\|f\|_{L^p(a, b)} < \infty$, where

$$\|f\|_{L^p(a, b)} = \begin{cases} (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \text{ess sup } |f(x)| : x \in [a, b], & \text{if } p = \infty. \end{cases}$$

The left sided fractional Riemann-Liouville integral of $f \in L^1(a, b)$ of order $\alpha > 0$ is given by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy,$$

for almost all $x \in (a, b)$, where $\Gamma(\alpha)$ is the standard Euler function. Similarly, the right sided fractional integral is defined, for almost all $x \in (a, b)$, as

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy,$$

where $(-1)^{-\alpha} = e^{-i\pi\alpha}$. If we consider the fractional integral I_{a+}^{α} (or I_{b-}^{α}) as an operator with domain $L^p(a, b)$, then its range is denoted by $I_{a+}^{\alpha}(L^p(a, b))$ (or $I_{b-}^{\alpha}(L^p(a, b))$). Clearly, for $\alpha = 1$, I_{a+}^{α} is the standard left integral operator, and a simple calculation yields that $\lim_{\alpha \rightarrow 0} (I_{a+}^{\alpha} f)(x) = f(x-) = \lim_{\varepsilon \downarrow 0} f(x-\varepsilon)$, for each $x \in (a, b)$. An immediate consequence of the definition of the fractional integral is that

$$I_{a+}^{\alpha} (I_{a+}^{\beta} f) = I_{a+}^{\alpha+\beta} f, \quad (4)$$

for all $\alpha, \beta > 0$. With some obvious variations where needed, all hold also for right sided fractional integrals; viz.

$$\begin{aligned} (I_{b-}^1 f)(x) &= (-1) \int_x^b f(y) dy, \\ \lim_{\alpha \rightarrow 0} (I_{b-}^{\alpha} f)(x) &= f(x+) = \lim_{\varepsilon \downarrow 0} f(x+\varepsilon), \\ I_{b-}^{\alpha} (I_{b-}^{\beta} f) &= I_{b-}^{\alpha+\beta} f, \quad \forall \alpha, \beta > 0 \end{aligned}$$

(See [20] for these and more on fractional calculus.)

Having defined a fractional integral, we now define a fractional derivative as the inverse of the fractional integral operator, whenever it is well defined. In other words, to each element f in $I_{a+}^{\alpha}(L^p(a, b))$ there corresponds a $\phi \in L^p(a, b)$, such that $I_{a+}^{\alpha} \phi = f$. This ϕ is unique in $L^p(a, b)$ and agrees almost everywhere with the fractional derivative, known as the left sided Riemann-Liouville or Weyl derivative, of α^{th} -order and defined as

$$\begin{aligned} D_{a+}^{\alpha} f(x) &= \left(\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^{\alpha}} dy \right) 1_{(a,b)}(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{1+\alpha}} dy \right) 1_{(a,b)}(x). \end{aligned} \quad (5)$$

Equivalently, we can write $D_{a+}^{\alpha} f = D(I_{a+}^{1-\alpha} f)$, where D is the standard derivative operator. Similarly, we can define the right sided Weyl derivative as $D_{b-}^{\alpha} f = D(I_{b-}^{1-\alpha} f)$, for which

$$\begin{aligned} D_{b-}^{\alpha} f(x) &= \left(\frac{(-1)^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^{\alpha}} dy \right) 1_{(a,b)}(x) \\ &= \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{1+\alpha}} dy \right) 1_{(a,b)}(x). \end{aligned} \quad (6)$$

As in the case of the integral operators, there is an analogue of the composition formula, given, for all $\alpha, \beta > 0$, by

$$D_{a+}^\alpha (D_{a+}^\beta f) = D_{a+}^{\alpha+\beta} f. \quad (7)$$

A similar formula also holds for the right sided derivatives, and is given by,

$$D_{b-}^\alpha (D_{b-}^\beta f) = D_{b-}^{\alpha+\beta} f, \quad (8)$$

as long as all the fractional derivatives are well defined.

We note that the linear spaces $I_{a+}^\alpha(L^p(a, b))$, for various choices of α and p , are Banach spaces equipped with the norms

$$\|f\|_{I_{a+}^\alpha(L^p(a, b))} = \|f\|_{L^p(a, b)} + \|D_{a+}^\alpha f\|_{L^p(a, b)},$$

and a similar norm is defined on the space $I_{b-}^\alpha(L^p(a, b))$.

Let $f(a+) = \lim_{\varepsilon \downarrow 0} f(a + \varepsilon)$, and $g(b-) = \lim_{\varepsilon \downarrow 0} g(b - \varepsilon)$, whenever the limit exists and is finite, and define

$$\begin{aligned} f_{a+}(x) &= (f(x) - f(a+))1_{(a, b)}(x), \\ g_{b-}(x) &= (g(x) - g(b-))1_{(a, b)}(x). \end{aligned}$$

Using the methods of fractional calculus (see [20]), an extension of the Stieltjes integral, called the *generalized Stieltjes integral*, of f with respect to g can be defined as

$$\int_a^b f(x) dg(x) = (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx, \quad (9)$$

where $f_{a+} \in I_{a+}^\alpha(L^p(a, b))$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q(a, b))$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$, and $\alpha p < 1$.

Next we define $C^\lambda(a, b; \mathbb{R}^d)$, the space of λ -Hölder continuous functions, with $\lambda \in (0, 1]$, as the space of \mathbb{R}^d valued functions for some fixed $d \in \mathbb{N}$, the set of natural numbers, equipped with the norm given by

$$\|f\|_{\lambda, a, b} := \sup_{a \leq c \leq d \leq b} \frac{\|f(d) - f(c)\|_2}{|d - c|^\lambda}, \quad (10)$$

where $\|\cdot\|_2$ is the usual Euclidean norm in the appropriate dimension. (When $a = 0$, we shall write $\|f\|_{\lambda, b}$ for $\|f\|_{\lambda, 0, b}$.)

In [20], Zähle proved that the conditions of the definition (9) are met if $f \in C^\lambda(0, T; \mathbb{R})$ and $g \in C^\mu(0, T; \mathbb{R})$ for $\lambda + \mu > 1$, in which case the integral defined in (9) coincides with the Riemann-Stieltjes integral. Now we state the following well known result concerning the Hölder coefficient and exponent of fractional Brownian motion with Hurst parameter H .

Lemma 2.1 *For $\{B^H(t) : t \in [0, T]\}$, a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, there exists, for each $0 < \varepsilon < H$ and $T > 0$, a positive random variable $\eta_{\varepsilon, T}$, such that $E(|\eta_{\varepsilon, T}|^p) < \infty$ for all $p \in [1, \infty)$ and, for all $s, t \in [0, T]$,*

$$|B^H(t) - B^H(s)| \leq \eta_{\varepsilon, T} |t - s|^{H-\varepsilon} \quad a.s.,$$

where $\eta_{\varepsilon, T} = C_{H, \varepsilon} T^{H-\varepsilon} \xi_T$, with the $L^q(\Omega)$ norm of ξ_T bounded by $c_{\varepsilon, q} T^\varepsilon$ for $q \geq \frac{2}{\varepsilon}$.

(For a proof of this, which involves a simple application of a Garsia-Rodemich-Rumsey type inequality, we refer the reader to [16].)

Consequently, using the above theory of deterministic fractional integration, integrals with respect to the fractional Brownian motion can also be defined, for appropriate integrands. This was done in [20], where a corresponding stochastic calculus is also developed with an appropriate change of variables formula.

For the following definitions, we shall assume $\alpha < \frac{1}{2}$.

Define $W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$ to be the space of measurable functions $g : [0,T] \rightarrow \mathbb{R}$, endowed with and finite under the norm

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0 < s < t < T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{|t-s|^{2-\alpha}} dy \right). \quad (11)$$

Clearly,

$$C^{1-\alpha+\varepsilon}(0,T;\mathbb{R}) \subset W_T^{1-\alpha,\infty}(0,T;\mathbb{R}) \subset C^{1-\alpha}(0,T;\mathbb{R}), \quad (12)$$

for all $\varepsilon > 0$. Moreover, if $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R})$, then $g|_{(0,t)} \in I_{t-}^{1-\alpha}(L^\infty(0,t))$ for all $t \in (0,T)$.

Recalling now the vector fields introduced in (3), the time has come to demand a set of regularity assumptions. Assume that there exist constants M_γ , $M_\gamma^{(1)}$ and $M_\gamma^{(2)}$ for all $\gamma \in \mathbb{N}$ such that:

- (A1) $|U_\gamma^i(x)| \leq M_\gamma$, $\forall x \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where U_γ^i denotes the i -th component of U_γ .
- (A2) $|U_\gamma^i(x) - U_\gamma^i(y)| \leq M_\gamma^{(1)} \|x - y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where $\|\cdot\|_2$ denotes the standard Euclidean norm in the appropriate dimension.
- (A3) $|W_{\gamma,j}^i(x) - W_{\gamma,j}^i(y)| \leq M_\gamma^{(2)} \|x - y\|_2$, $\forall x, y \in \mathbb{R}^n$ and $\gamma \in \mathbb{N}$, where $W_\gamma(x)$ denotes the spatial derivative of $U_\gamma(x)$, and $W_{\gamma,j}^i(\cdot)$ denotes the (i,j) -th element of the matrix $W_\gamma(\cdot)$.
- (A4) $M^{(1)} = \sum_{\alpha \in \mathbb{N}} M_\alpha^{(1)} < \infty$, $M^{(2)} = \sum_{\alpha \in \mathbb{N}} M_\alpha^{(2)} < \infty$, and $M^{(3)} = \sum_{\alpha \in \mathbb{N}} M_\alpha^{(3)} < \infty$.

Under conditions (A1) – (A4), existence and uniqueness of the solution of (3), in the space $C^{1-\alpha}(0,T;\mathbb{R}^n)$ is proven in [16] for $|I| < \infty$.

In fact, the existence and uniqueness of the solution can be proven under far weaker conditions, but without necessarily giving a solution which provides a diffeomorphism in \mathbb{R}^n (cf. [9] for details). Properties of the solution of the flow equation are also obtained in [16], and improved on in [17].

3 The main result

We shall now adopt and adapt the approach developed in [17] to derive some estimates on some of the basic geometric characteristics of the flow (3).

With M , as usual, a C^2 , m -dimensional manifold embedded in \mathbb{R}^n , we write $T_x M$ for its tangent space at x . Let $v \in T_x M$. Then its push-forward under the flow Φ_t is denoted by

$$v_t = D\Phi_t(x)v,$$

where $D\Phi_t(x) = (\frac{\partial \Phi_t^i(x)}{\partial x^j})_{ij}$ denotes the matrix of spatial derivatives of the flow $\Phi_t(x)$, and $v_t \in T_{x_t} M_t$. From now on we shall write x_t for $\Phi_t(x)$.

We now prove the following technical result, which will form the basis for much of the subsequent analysis.

Theorem 3.1 *Under assumptions (A1) – (A4), and for $\alpha = 1 - H + \delta$, $\beta = H - \varepsilon$, such that $(1 - H) < \alpha < 1/2$ and $\delta > \varepsilon$, there exist a constant c and a random variable C_T , such that*

$$\begin{aligned} \sup_{r \in [0, T]} \|v_r\|_2 &\leq \sup_{r \in [0, T]} \|v_r\|_1 \\ &\leq c 2^{C_T T}, \end{aligned}$$

where $\|v_r\|_2$ and $\|v_r\|_1$ denote the l_2 and l_1 norms, respectively, of the vector v_r as an element in \mathbb{R}^n . The random variable C_T depends on α , β , n , I , and $\{\|B_\gamma^H\|_{\beta, T}, M_\gamma, M_\gamma^{(1)}, M_\gamma^{(2)}\}_{\gamma \in I}$. Furthermore,

$$E[C_T]^\beta \leq C \cdot E[\|B^H\|_{\beta, T}],$$

where the constant C depends only on α , β , n , $|I|$ and $\{M_\gamma, M_\gamma^{(1)}, M_\gamma^{(2)}\}_{\gamma \in I}$.

Remark 3.2 *For a better understanding of the results of Theorem 3.1, we note that for the case $|I| = 1$, this result simplifies to*

$$\sup_{r \in [0, T]} \|v_r\|_2 \leq c 2^{C T} \|B^H\|_{\beta, T}^{1/\beta},$$

for some constants c and C , dependent only on the various uniform bounds and the Lipschitz coefficients corresponding to the vector field.

Remark 3.3 *The results listed in this section hold true for any $I \subset \mathbb{N}$ as long as the cardinality of the set satisfies $|I| < \infty$. However, extensions of these results to the case $I = \mathbb{N}$, though possible, require unnatural conditions on the summability of the constants appearing in Assumptions (A1) – (A3). For instance, extending Lemma 3.1 to the case $I = \mathbb{N}$ would require*

$$\frac{\sum_{\gamma \in \mathbb{N}} M_\gamma^{(1)} \|B_\gamma^H\|_{\beta, T}}{\sum_{\gamma \in \mathbb{N}} M_\gamma^{(2)} \|B_\gamma^H\|_{\beta, T}} < \infty.$$

This, in turn would be implied by $\sum_{\gamma \in \mathbb{N}} M_\gamma^{(1)} / M_\gamma^{(2)} < \infty$, which does not seem to have a clear meaning in terms of the vector fields U_γ .

The idea of the proof of Theorem 3.1 is to break up the interval $[0, T]$ into smaller units of size Δ , on which reasonable estimates of $\|v_r\|_2$ are possible, and then to glue the intervals together to obtain the required result. However, in the process, we shall need to derive an estimate on the flow, presented in the following lemma, the proof of which relies on some results of [17].

Lemma 3.1 *Let $M, M^{(1)}$ be constants as defined in Assumptions (A1) – (A4), and $0 \leq s \leq t \leq T$ be such that*

$$(t-s)^{-\beta} > \frac{n\alpha(2\alpha+\beta-1)}{2(1-\alpha)(1-2\alpha)(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{\gamma \in I} M_{\gamma}^{(1)} \|B_{\gamma}^H\|_{\beta, T},$$

where $\alpha = 1 - H + \delta$, $\beta = H - \varepsilon$, such that $(1 - H) < \alpha < 1/2$ and $\delta > \varepsilon$. Then for x_t defined in (3) there exists a positive random variable $K_{s,t}^$ such that*

$$\int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1+\alpha}} dr \leq K_{s,t}^* (t-s)^{\beta-\alpha}. \quad (13)$$

Furthermore, $K_{s,t}^$ can be bounded above by another random variable, independent of s and t , with finite moments of order greater than 1, as long as $(t-s)$ is chosen sufficiently small.*

Remark 3.4 *Note that under the aforementioned conditions concerning α and β , we have $\alpha + \beta > 1$, and $\beta > \alpha$.*

Proof: Writing $U_{\gamma}^i(\cdot)$ for the i -th component of the vector $U_{\gamma}(\cdot)$ and choosing $\{e_i\}_{i=1}^n$ as the canonical basis of \mathbb{R}^n , we have

$$\langle (x_t - x_s), e_i \rangle = \sum_{\gamma \in I} \int_s^t U_{\gamma}^i(x_r) dB_{\gamma}^H(r),$$

which is true by linearity of the operation, and where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Hence for $\alpha \in (1 - H, \frac{1}{2})$, using (9), we obtain

$$\begin{aligned} |\langle (x_t - x_s), e_i \rangle| &= \left| \sum_{\gamma \in I} \int_s^t U_{\gamma}^i(x_r) dB_{\gamma}^H(r) \right| \\ &= \left| \sum_{\gamma \in I} \int_s^t D_{s+}^{\alpha} U_{\gamma}^i(x_r) D_{t-}^{1-\alpha} B_{\gamma, t-}^H(r) dr \right| \\ &\leq \sum_{\gamma \in I} \int_s^t |D_{s+}^{\alpha} U_{\gamma}^i(x_r)| \cdot |D_{t-}^{1-\alpha} B_{\gamma, t-}^H(r)| dr \end{aligned}$$

To obtain a bound on the second term in the integrand, choose $\beta < H$, such that $\alpha + \beta > 1$,

so that using (6), we have

$$\begin{aligned}
|D_{t-}^{1-\alpha} B_{\gamma,t-}^H(r)| &= \left| \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{B_{\gamma}^H(t) - B_{\gamma}^H(r)}{(t-r)^{1-\alpha}} + \alpha \int_r^t \frac{B_{\gamma}^H(u) - B_{\gamma}^H(r)}{(u-r)^{2-\alpha}} du \right) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{|B_{\gamma}^H(t) - B_{\gamma}^H(r)|}{|t-r|^{1-\alpha}} + \alpha \int_r^t \frac{|B_{\gamma}^H(u) - B_{\gamma}^H(r)|}{(u-r)^{2-\alpha}} du \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(\frac{|B_{\gamma}^H(t) - B_{\gamma}^H(r)|(t-r)^{\beta}}{(t-r)^{\beta}(t-r)^{1-\alpha}} \right. \\
&\quad \left. + \alpha \int_r^t \frac{|B_{\gamma}^H(u) - B_{\gamma}^H(r)|}{(u-r)^{\beta}} (u-r)^{\alpha+\beta-2} du \right) \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\|B_{\gamma}^H\|_{\beta,T} (t-r)^{\alpha+\beta-1} + \alpha \|B_{\gamma}^H\|_{\beta,T} \frac{(t-r)^{\alpha+\beta-1}}{\alpha+\beta-1} \right) \\
&= k_1(\alpha, \beta) \|B_{\gamma}^H\|_{\beta,T} (t-r)^{\alpha+\beta-1}, \tag{14}
\end{aligned}$$

where $k_1(\alpha, \beta) = \frac{(2\alpha+\beta-1)}{(\alpha+\beta-1)\Gamma(\alpha)}$.

To bound the first term we use (5) and assumptions (A1) – (A2) to see that

$$\begin{aligned}
|D_{s+}^{\alpha} U_{\gamma}^i(x_r)| &= \frac{1}{\Gamma(1-\alpha)} \left| \frac{U_{\gamma}^i(x_r)}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{U_{\gamma}^i(x_r) - U_{\gamma}^i(x_{\theta})}{(r-\theta)^{1+\alpha}} d\theta \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \left(\frac{|U_{\gamma}^i(x_r)|}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{|U_{\gamma}^i(x_r) - U_{\gamma}^i(x_{\theta})|}{(r-\theta)^{1+\alpha}} d\theta \right) \\
&\leq c_{\alpha} \left(\frac{M_{\gamma}}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{M_{\gamma}^{(1)} \|x_r - x_{\theta}\|_2}{(r-\theta)^{1+\alpha}} d\theta \right) \\
&\leq c_{\alpha} \left(M_{\gamma} (r-s)^{-\alpha} + M_{\gamma,\alpha}^{(1)} \|x\|_{s,r,1-\alpha} (r-s)^{1-2\alpha} \right), \tag{15}
\end{aligned}$$

where $c_{\alpha} = \Gamma(1-\alpha)^{-1}$, $M_{\gamma,\alpha}^{(1)} = \frac{\alpha M_{\gamma}^{(1)}}{(1-2\alpha)}$ and $\|x\|_{s,r,1-\alpha}$ is the Hölder norm as defined in (10).

Therefore, combining the above two estimates, we find

$$\begin{aligned}
|\langle (x_t - x_s), e_i \rangle| &\leq c_{\alpha} k_1(\alpha, \beta) \sum_{\gamma \in I} \|B_{\gamma}^H\|_{\beta,T} \int_s^t \left(M_{\gamma} (r-s)^{-\alpha} (t-r)^{\alpha+\beta-1} \right. \\
&\quad \left. + M_{\gamma,\alpha}^{(1)} \|x\|_{s,r,1-\alpha} (r-s)^{1-2\alpha} (t-r)^{\alpha+\beta-1} \right) dr \\
&\leq c_{\alpha} k_1(\alpha, \beta) \sum_{\gamma \in I} \|B_{\gamma}^H\|_{\beta,T} (t-s)^{\alpha+\beta-1} \int_s^t \left(M_{\gamma} (r-s)^{-\alpha} \right. \\
&\quad \left. + M_{\gamma,\alpha}^{(1)} \|x\|_{s,r,1-\alpha} (r-s)^{1-2\alpha} \right) dr \\
&\leq c_{\alpha} k_1(\alpha, \beta) \sum_{\gamma \in I} \|B_{\gamma}^H\|_{\beta,T} \left(M_{\gamma} (t-s)^{\beta} (1-\alpha)^{-1} \right. \\
&\quad \left. + M_{\gamma,\alpha}^{(1)} \|x\|_{s,t,1-\alpha} (t-s)^{1-\alpha+\beta} (2-2\alpha)^{-1} \right).
\end{aligned}$$

Let

$$M_\alpha = (1 - \alpha)^{-1} \sum_{\gamma \in I} M_\gamma \|B_\gamma^H\|_{\beta, T}, \quad (16)$$

and

$$\tilde{M}_\alpha^{(1)} = (2 - 2\alpha)^{-1} \sum_{\gamma \in I} M_{\gamma, \alpha}^{(1)} \|B_\gamma^H\|_{\beta, T}. \quad (17)$$

Then

$$\begin{aligned} \|(x_t - x_s)\|_1 &= \sum_{i=1}^n |\langle (x_t - x_s), e_i \rangle| \\ &\leq c_\alpha n k_1(\alpha, \beta) \left(M_\alpha (t - s)^\beta + \tilde{M}_\alpha^{(1)} \|x\|_{s, t, 1-\alpha} (t - s)^{1-\alpha+\beta} \right). \end{aligned}$$

Equivalently,

$$\begin{aligned} \frac{\|(x_t - x_s)\|_1}{(t - s)^{1-\alpha}} &\leq c_\alpha n k_1(\alpha, \beta) \left(M_\alpha (t - s)^{\alpha+\beta-1} \right. \\ &\quad \left. + \tilde{M}_\alpha^{(1)} \|x\|_{s, t, 1-\alpha} (t - s)^\beta \right). \end{aligned} \quad (18)$$

(Recall that $\alpha + \beta > 1$.)

Now using the above estimate, and the fact that $\|\cdot\|_2$ is bounded above by $\|\cdot\|_1$, we have

$$\begin{aligned} \|x\|_{s, t, 1-\alpha} &= \sup_{s \leq u \leq v \leq t} \frac{\|(x_v - x_u)\|_2}{(v - u)^{1-\alpha}} \\ &\leq \sup_{s \leq u \leq v \leq t} \frac{\|(x_v - x_u)\|_1}{(v - u)^{1-\alpha}} \\ &\leq \sup_{s \leq u \leq v \leq t} c_\alpha n k_1(\alpha, \beta) \left(M_\alpha (v - u)^{\alpha+\beta-1} \right. \\ &\quad \left. + \tilde{M}_\alpha^{(1)} \|x\|_{u, v, 1-\alpha} (v - u)^\beta \right) \\ &\leq c_\alpha n k_1(\alpha, \beta) \left(M_\alpha (t - s)^{\alpha+\beta-1} \right. \\ &\quad \left. + \tilde{M}_\alpha^{(1)} \|x\|_{s, t, 1-\alpha} (t - s)^\beta \right). \end{aligned} \quad (19)$$

Now choosing s, t such that

$$(t - s)^{-\beta} > c_\alpha n k_1(\alpha, \beta) \tilde{M}_\alpha^{(1)}, \quad (20)$$

(19) can be rewritten as

$$\begin{aligned} \|x\|_{s, t, 1-\alpha} &\leq \frac{c_\alpha n k_1(\alpha, \beta) M_\alpha (t - s)^{\alpha+\beta-1}}{1 - c_\alpha n k_1(\alpha, \beta) \tilde{M}_\alpha^{(1)} (t - s)^\beta} \\ &= K_{s, t} (t - s)^{\alpha+\beta-1}, \end{aligned} \quad (21)$$

where $K_{s,t} = \frac{c_\alpha n k_1(\alpha, \beta) M_\alpha}{1 - c_\alpha n k_1(\alpha, \beta) M_\alpha^{(1)}(t-s)^\beta}$.

Therefore,

$$\begin{aligned} \int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1+\alpha}} dr &= \int_s^t \frac{\|x_t - x_r\|_2}{(t-r)^{1-\alpha}} (t-r)^{-2\alpha} dr \\ &\leq \|x\|_{s,t,1-\alpha} \int_s^t (t-r)^{-2\alpha} dr \\ &\leq K_{s,t} \frac{(t-s)^{\beta-\alpha}}{(1-2\alpha)} \\ &= K_{s,t}^* (t-s)^{\beta-\alpha}, \end{aligned}$$

where $K_{s,t}^* = \frac{K_{s,t}}{(1-2\alpha)}$, thus establishing (13). The final claim, that $K_{s,t}^*$ can be bounded by a random variable independent of s and t , will be proven later.

Proof of Theorem 3.1: Taking the space derivative of (3), the existence of which is ensured by Theorem 3.2 in [17], we have

$$D\Phi_t(x) = I + \sum_{\gamma \in I} \int_0^t W_\gamma(\Phi_s(x)) D\Phi_s(x) dB_\gamma^H(s),$$

where the matrix $W_\gamma(\cdot) = (W_{\gamma,j}^i(\cdot))_{i,j}$ denotes the spatial derivative of the vector field U .

Now using the definition of the pushforward of a vector, we can write the evolution equation of the tangent vector as follows

$$v_t = v + \sum_{\gamma \in I} \int_0^t W_\gamma(x_s) v_s dB_\gamma^H(s).$$

Recall that $\|v_t\|_1 = \sum_{i=1}^n |\langle v_t, e_i \rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product, and $\{e_i\}_{i=1}^n$ denotes the canonical basis of \mathbb{R}^n . Since,

$$\langle v_t, e_i \rangle = x + \sum_{\gamma \in I} \int_0^t \langle W_\gamma(x_r) v_r, e_i \rangle dB_\gamma^H(r),$$

we have

$$\begin{aligned} |\langle v_t, e_i \rangle - \langle v_s, e_i \rangle| &= \left| \sum_{\gamma \in I} \int_s^t \langle W_\gamma(x_r) v_r, e_i \rangle dB_\gamma^H(r) \right| \\ &= \left| \sum_{\gamma \in I} \int_s^t D_{s+}^\alpha \langle W_\gamma(x_r) v_r, e_i \rangle D_{t-}^{1-\alpha} B_{\gamma,t-}^H(r) dr \right| \\ &\leq \sum_{\gamma \in I} \int_s^t |D_{s+}^\alpha \langle W_\gamma(x_r) v_r, e_i \rangle| \cdot |D_{t-}^{1-\alpha} B_{\gamma,t-}^H(r)| dr. \end{aligned}$$

The above inequality holds for any choice of s and t , but we are interested in pairs for which $(t - s)$ is sufficiently small. To this end, note first that from (14) we can bound the second integrand by

$$|D_{t-}^{1-\alpha} B_{\gamma, t-}^H(r)| \leq k_1(\alpha, \beta) \|B_{\gamma}^H\|_{\beta, T} (t - r)^{\alpha+\beta-1}.$$

Now using (5) and Assumptions (A2)–(A4), the first integrand can be bounded by

$$\begin{aligned} |D_{s+}^{\alpha} \langle W_{\gamma}(x_r) v_r, e_i \rangle| &\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{|\langle W_{\gamma}(x_r) v_r, e_i \rangle|}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{|\langle W_{\gamma}(x_r) v_r, e_i \rangle - \langle W_{\gamma}(x_{\theta}) v_{\theta}, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{\sum_{j=1}^n |W_{\gamma, j}^i(x_r) \langle v_r, e_j \rangle|}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{|\langle W_{\gamma}(x_r) v_r, e_i \rangle - \langle W_{\gamma}(x_{\theta}) v_{\theta}, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[M_{\gamma}^{(1)} \sum_{j=1}^n \frac{|\langle v_r, e_j \rangle|}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{|\langle W_{\gamma}(x_r) v_r, e_i \rangle - \langle W_{\gamma}(x_{\theta}) v_r, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right. \\ &\quad \left. + \alpha \int_s^r \frac{|\langle W_{\gamma}(x_{\theta}) v_r, e_i \rangle - \langle W_{\gamma}(x_{\theta}) v_{\theta}, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \left[M_{\gamma}^{(1)} \sum_{j=1}^n \frac{|\langle v_r, e_j \rangle|}{(r-s)^{\alpha}} \right. \\ &\quad \left. + \alpha \int_s^r \frac{|\sum_{j=1}^n (W_{\gamma, j}^i(x_r) \langle v_r, e_j \rangle - W_{\gamma, j}^i(x_{\theta}) \langle v_r, e_j \rangle)|}{(r-\theta)^{1+\alpha}} d\theta \right. \\ &\quad \left. + \alpha \int_s^r \frac{|\sum_{j=1}^n (W_{\gamma, j}^i(x_{\theta}) \langle v_r, e_j \rangle - W_{\gamma, j}^i(x_{\theta}) \langle v_{\theta}, e_i \rangle)|}{(r-\theta)^{1+\alpha}} d\theta \right] \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[M_{\gamma}^{(1)} \sum_{j=1}^n \frac{|\langle v_r, e_j \rangle|}{(r-s)^{\alpha}} \right. \\ &\quad \left. + \alpha \int_s^r \frac{\sum_{j=1}^n |W_{\gamma, j}^i(x_r) - W_{\gamma, j}^i(x_{\theta})| \cdot |\langle v_r, e_j \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right. \\ &\quad \left. + \alpha \int_s^r \frac{\sum_{j=1}^n |W_{\gamma, j}^i(x_{\theta})| \cdot |\langle v_r, e_j \rangle - \langle v_{\theta}, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left[M_{\gamma}^{(1)} \sum_{j=1}^n \frac{|\langle v_r, e_j \rangle|}{(r-s)^{\alpha}} + \alpha M_{\gamma}^{(2)} \sum_{j=1}^n |\langle v_r, e_j \rangle| \int_s^r \frac{\|x_r - x_{\theta}\|_2}{(r-\theta)^{1+\alpha}} d\theta \right. \\ &\quad \left. + \alpha M_{\gamma}^{(1)} \sum_{j=1}^n \int_s^r \frac{|\langle v_r, e_j \rangle - \langle v_{\theta}, e_i \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right]. \end{aligned}$$

Now using the result proven in Lemma 3.1, for r such that $s < r < t$, with $(t - s)$ satisfying (20), we have

$$\int_s^r \frac{\|x_r - x_{\theta}\|_2}{(r-\theta)^{1+\alpha}} d\theta \leq K_{s, r}^* (r - s)^{\beta-\alpha}.$$

Hence,

$$\begin{aligned}
|D_{s+}^\alpha \langle W_\gamma(x_r) v_r, e_i \rangle| &\leq \sum_{j=1}^n \left[\frac{|\langle v_r, e_j \rangle|}{(r-s)^\alpha} \left(\frac{M_\gamma^{(1)} + \alpha M_\gamma^{(2)} K_{s,r}^* (r-s)^\beta}{\Gamma(1-\alpha)} \right) \right. \\
&\quad \left. + \frac{\alpha M_\gamma^{(1)}}{\Gamma(1-\alpha)} \int_s^r \frac{|\langle v_r, e_j \rangle - \langle v_\theta, e_j \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\
&= \sum_{j=1}^n \left[a_{\gamma,s,r,1} \frac{|\langle v_r, e_j \rangle|}{(r-s)^\alpha} + \frac{\alpha M_\gamma^{(1)}}{\Gamma(1-\alpha)} \int_s^r \frac{|\langle v_r, e_j \rangle - \langle v_\theta, e_j \rangle|}{(r-\theta)^{1+\alpha}} d\theta \right] \\
&\leq \sum_{j=1}^n \left[a_{\gamma,s,r,1} |\langle v_r, e_j \rangle| (r-s)^{-\alpha} + b_{\gamma,1} \|\langle v, e_j \rangle\|_{s,t,\beta} (r-s)^{\beta-\alpha} \right],
\end{aligned}$$

where

$$a_{\gamma,s,r,1} = \frac{M_\gamma^{(1)} + \alpha M_\gamma^{(2)} K_{s,r}^* (r-s)^\beta}{\Gamma(1-\alpha)}, \quad (22)$$

and

$$b_{\gamma,1} = \frac{\alpha M_\gamma^{(1)}}{(\beta-\alpha)\Gamma(1-\alpha)}. \quad (23)$$

Note that $a_{\gamma,s,r,1} \leq a_{\gamma,s,t,1}$, for $s \leq r \leq t$.

Writing $a_{s,r,1} = \sum_{\gamma \in I} a_{\gamma,s,r,1} \|B_\gamma^H\|_{\beta,T}$ and $b_1 = \sum_{\gamma \in I} b_{\gamma,1} \|B_\gamma^H\|_{\beta,T}$, and using the above

estimates for the integrands, together with (14) and Remark 3.4, we have

$$\begin{aligned}
|\langle (v_t - v_s), e_i \rangle| &\leq k_1(\alpha, \beta) \int_s^t \sum_{j=1}^n \left(a_{s,r,1} |\langle v_r, e_j \rangle| (r-s)^{-\alpha} (t-r)^{\alpha+\beta-1} \right. \\
&\quad \left. + b_1 \|\langle v, e_j \rangle\|_{s,t,\beta} (r-s)^{\beta-\alpha} (t-r)^{\alpha+\beta-1} \right) dr \\
&\leq k_1(\alpha, \beta) (t-s)^{\alpha+\beta-1} \int_s^t \sum_{j=1}^n \left(a_{s,r,1} |\langle v_r, e_j \rangle| (r-s)^{-\alpha} \right. \\
&\quad \left. + b_1 \|\langle v, e_j \rangle\|_{s,t,\beta} (r-s)^{\beta-\alpha} \right) dr \\
&\leq k_1(\alpha, \beta) (t-s)^{\alpha+\beta-1} \int_s^t \sum_{j=1}^n \left(a_{s,r,1} \|\langle v, e_j \rangle\|_{s,t,\infty} (r-s)^{-\alpha} \right. \\
&\quad \left. + b_1 \|\langle v, e_j \rangle\|_{s,t,\beta} (r-s)^{\beta-\alpha} \right) dr \\
&\leq k_1(\alpha, \beta) (t-s)^{\alpha+\beta-1} \sum_{j=1}^n \left(a_{s,t,1} \|\langle v, e_j \rangle\|_{s,t,\infty} \frac{(t-s)^{1-\alpha}}{1-\alpha} \right. \\
&\quad \left. + b_1 \|\langle v, e_j \rangle\|_{s,t,\beta} \frac{(t-s)^{1+\beta-\alpha}}{1+\beta-\alpha} \right) \\
&= k_1(\alpha, \beta) \sum_{j=1}^n \left(a_{s,t,2} \|\langle v, e_j \rangle\|_{s,t,\infty} (t-s)^\beta \right. \\
&\quad \left. + b_2 \|\langle v, e_j \rangle\|_{s,t,\beta} (t-s)^{2\beta} \right),
\end{aligned}$$

where $a_{s,t,2} = a_{s,t,1}(1-\alpha)^{-1}$ and $b_2 = b_1(1-\alpha+\beta)^{-1}$.

Therefore,

$$\begin{aligned}
\|\langle v, e_i \rangle\|_{s,t,\beta} &= \sup_{s \leq r \leq \theta \leq t} \frac{|\langle (v_\theta - v_r), e_i \rangle|}{(\theta-r)^\beta} \\
&\leq k_1(\alpha, \beta) \sum_{j=1}^n \sup_{s \leq r \leq \theta \leq t} \left(a_{r,\theta,2} \|\langle v, e_j \rangle\|_{r,\theta,\infty} \right. \\
&\quad \left. + b_2 \|\langle v, e_j \rangle\|_{r,\theta,\beta} (\theta-r)^\beta \right) \\
&\leq k_1(\alpha, \beta) \sum_{j=1}^n \left(a_{s,t,2} \|\langle v, e_j \rangle\|_{s,t,\infty} \right. \\
&\quad \left. + b_2 \|\langle v, e_j \rangle\|_{s,t,\beta} (t-s)^\beta \right).
\end{aligned}$$

As a consequence of the above estimate we have

$$\begin{aligned}
\sum_{i=1}^n \|\langle v, e_i \rangle\|_{s,t,\beta} &\leq n k_1(\alpha, \beta) \sum_{j=1}^n \left(a_{s,t,2} \|\langle v, e_j \rangle\|_{s,t,\infty} \right. \\
&\quad \left. + b_2 \|\langle v, e_j \rangle\|_{s,t,\beta} (t-s)^\beta \right). \tag{24}
\end{aligned}$$

For further analysis we shall require that

$$(t-s)^{-\beta} > n k_1(\alpha, \beta) b_2. \quad (25)$$

Thereby, for $(t-s)$ satisfying conditions (20) and (25), we can rewrite (24) as

$$\sum_{i=1}^n \|\langle v, e_i \rangle\|_{s,t,\beta} \leq n k_1(\alpha, \beta) a_{s,t,2} \sum_{i=1}^n \frac{\|\langle v, e_i \rangle\|_{s,t,\infty}}{(1 - n k_1(\alpha, \beta) b_2 (t-s)^\beta)}.$$

Hence,

$$\begin{aligned} \sum_{i=1}^n |\langle v_t, e_i \rangle| &\leq \sum_{i=1}^n \left(|\langle v_s, e_i \rangle| + |\langle v_t, e_i \rangle - \langle v_s, e_i \rangle| \right) \\ &\leq \sum_{i=1}^n \left(|\langle v_s, e_i \rangle| + \|\langle v, e_i \rangle\|_{s,t,\beta} (t-s)^\beta \right) \\ &\leq \sum_{i=1}^n \left(|\langle v_s, e_i \rangle| + n k_1(\alpha, \beta) a_{s,t,2} \frac{\|\langle v, e_i \rangle\|_{s,t,\infty} (t-s)^\beta}{(1 - n k_1(\alpha, \beta) b_2 (t-s)^\beta)} \right) \end{aligned}$$

Clearly, for any $r \in [s, t]$ we have

$$\sum_{i=1}^n |\langle v_r, e_i \rangle| \leq \sum_{i=1}^n \left(|\langle v_s, e_i \rangle| + n k_1(\alpha, \beta) a_{s,r,2} \frac{\|\langle v, e_i \rangle\|_{s,r,\infty} (r-s)^\beta}{(1 - n k_1(\alpha, \beta) b_2 (r-s)^\beta)} \right).$$

Now using the fact that $s < r < t$, so that $\|\langle v, e_i \rangle\|_{s,r,\infty} \leq \|\langle v, e_i \rangle\|_{s,t,\infty}$ and $a_{s,r,2} \leq a_{s,t,2}$, we have

$$\begin{aligned} \sum_{i=1}^n \|\langle v, e_i \rangle\|_{s,t,\infty} &\leq \sum_{i=1}^n \left(|\langle v_s, e_i \rangle| \right. \\ &\quad \left. + n k_1(\alpha, \beta) a_{s,t,2} \frac{\|\langle v, e_i \rangle\|_{s,t,\infty} (t-s)^\beta}{(1 - n k_1(\alpha, \beta) b_2 (t-s)^\beta)} \right). \end{aligned} \quad (26)$$

Finally, we shall require $(t-s)$ to satisfy

$$(t-s)^{-\beta} > n k_1(\alpha, \beta) [a_{s,t,2} + b_2], \quad (27)$$

to allow us to rewrite (26) as

$$\sum_{i=1}^n \|\langle v, e_i \rangle\|_{s,t,\infty} \left[1 - \frac{n k_1(\alpha, \beta) a_{s,t,2} (t-s)^\beta}{(1 - n k_1(\alpha, \beta) b_2 (t-s)^\beta)} \right] \leq \sum_{i=1}^n |\langle v_s, e_i \rangle|.$$

We shall note that for $(t-s)$ sufficiently small, the inequality (27) does hold true, as $a_{s,t,2}$ is a decreasing function of $(t-s)$.

This, in turn implies,

$$\begin{aligned}
\sum_{i=1}^n \sup_{0 \leq r \leq t} |\langle v_r, e_i \rangle| &= \sum_{i=1}^n \max \left\{ \sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|, \|\langle v, e_i \rangle\|_{s,t,\infty} \right\} \\
&\leq \sum_{i=1}^n \max \left\{ \sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|, \frac{|\langle v_s, e_i \rangle|}{\left[1 - \frac{nk_1(\alpha, \beta) a_{s,t,2} (t-s)^\beta}{(1-nk_1(\alpha, \beta) b_2 (t-s)^\beta)}\right]} \right\} \\
&\leq \sum_{i=1}^n \max \left\{ \sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|, \frac{\sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|}{\left[1 - \frac{nk_1(\alpha, \beta) a_{s,t,2} (t-s)^\beta}{(1-nk_1(\alpha, \beta) b_2 (t-s)^\beta)}\right]} \right\} \\
&= \sum_{i=1}^n \frac{\sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|}{\left[1 - \frac{nk_1(\alpha, \beta) a_{s,t,2} (t-s)^\beta}{(1-nk_1(\alpha, \beta) b_2 (t-s)^\beta)}\right]} \\
&= S \sum_{i=1}^n \sup_{0 \leq r \leq s} |\langle v_r, e_i \rangle|, \tag{28}
\end{aligned}$$

where $S = \left[1 - \frac{nk_1(\alpha, \beta) a_{s,t,2} (t-s)^\beta}{(1-nk_1(\alpha, \beta) b_2 (t-s)^\beta)}\right]^{-1}$.

Next we divide the interval $[0, T]$ into p pieces of size $\Delta = (t - s)$, with Δ being small enough, so that none of the above estimates fail, and write $a_{\Delta,2}$ for $a_{s,t,2}$, as $a_{s,t,2}$ depends on s, t only through the difference $(t - s) = \Delta$.

More precisely, in view of (20), (25) and (27), we require Δ to satisfy

$$\begin{aligned}
\Delta^{-\beta} &> n k_1(\alpha, \beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)}, b_2, (a_{\Delta,2} + b_2)] \\
&= n k_1(\alpha, \beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)}, (a_{\Delta,2} + b_2)].
\end{aligned}$$

For example, we can choose

$$\Delta^{-\beta} = 3 n k_1(\alpha, \beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)}, (a_{\Delta,2} + b_2)], \tag{29}$$

and thus, for this specific choice of Δ , we have $S \leq 2$.

To ensure the existence of such a Δ , we start with

$$\Delta_0^{-\beta} = 3 n k_1(\alpha, \beta) c_\alpha \tilde{M}_\alpha^{(1)}.$$

Then, if

$$\Delta_0^{-\beta} \geq 3 n k_1(\alpha, \beta) (a_{\Delta_0,2} + b_2), \tag{30}$$

we choose $\Delta = \Delta_0$. Otherwise we solve the equation

$$\Delta^{-\beta} = 3 n k_1(\alpha, \beta) (a_{\Delta,2} + b_2),$$

in the range $\Delta \leq \Delta_0$. It is easy to see that the solution to this equation is ensured since the left side increases to infinity as $\Delta \rightarrow 0$, whereas the right side, which is larger than the left side at $\Delta = \Delta_0$, decreases as Δ decreases to zero.

Using the above notation, and repeatedly applying the technique used in (28), we can write

$$\begin{aligned}
\sup_{t \in [0, T]} \|v_t\|_1 &= \sup_{t \in [0, p\Delta]} \left[\sum_{i=1}^n |\langle v_t, e_i \rangle| \right] \\
&\leq \sum_{i=1}^n \sup_{t \in [0, p\Delta]} |\langle v_t, e_i \rangle| \\
&\leq S^p \sum_{i=1}^n |\langle v, e_i \rangle|,
\end{aligned}$$

where

$$\begin{aligned}
p = \frac{T}{\Delta} &= T \left(3n k_1(\alpha, \beta) \cdot \max[c_\alpha \tilde{M}_\alpha^{(1)}, (a_{\Delta,2} + b_2)] \right)^{1/\beta} \\
&= T C_T,
\end{aligned}$$

and

$$C_T = \left(3n k_1(\alpha, \beta) \cdot \max[(c_\alpha \tilde{M}_\alpha^{(1)}), (a_{\Delta,2} + b_2)] \right)^{1/\beta}.$$

Since we have all the appropriate notation at hand, we now take a moment off the proof of Theorem 3.1 to complete the remaining issues in the proof of Lemma 3.1

Proof of Lemma 3.1 (continued): To prove the final claim of Lemma 3.1, note that for specific choice $(t - s) = \Delta$, together with (16) and (21) we have

$$\begin{aligned}
K_{s,t}^* &= \frac{K_{s,t}}{(1 - 2\alpha)} \\
&\leq \frac{3}{2(1 - 2\alpha)} \cdot c_\alpha n k_1(\alpha, \beta) M_\alpha \\
&= \frac{3}{2(1 - 2\alpha)} \cdot c_\alpha n k_1(\alpha, \beta) \sum_{\gamma \in I} \frac{M_\gamma \|B_\gamma^H\|_{\beta, T}}{1 - \alpha},
\end{aligned}$$

and so there exists a constant $K(\alpha, \beta)$, dependent only on α and β , such that

$$\begin{aligned}
K_{s,t}^* (t - s)^\beta &\leq K(\alpha, \beta) \frac{\sum_{\gamma \in I} M_\gamma \|B_\gamma^H\|_{\beta, T}}{2 \sum_{\gamma \in I} M_\gamma^{(1)} \|B_\gamma^H\|_{\beta, T}} \\
&\leq K(\alpha, \beta) \sum_{\gamma \in I} \frac{M_\gamma}{M_\gamma^{(1)}}.
\end{aligned}$$

Consequently, $a_{\Delta,2}$ can also be bounded above by a constant a_2 , hence we shall replace $a_{\Delta,2}$ by a_2 , in the following discussion. \square

Returning to the proof of Theorem 3.1, note that

$$(C_T)^\beta \leq 3n c_\alpha k_1(\alpha, \beta) \sum_{\gamma \in I} (\tilde{M}_{\alpha, \gamma}^{(1)} + a_{2, \gamma} + b_{2, \gamma}) \|B_\gamma^H\|_{\beta, T},$$

where $\tilde{M}_{\alpha,\gamma}^{(1)}$, $a_{2,\gamma}$, and $b_{2,\gamma}$ are the coefficients of $\|B_\gamma^H\|_{\beta,T}$ in the constants $\tilde{M}_\alpha^{(1)}$, a_2 and b_2 , respectively.

Now using the bound on S available due to the specific choice of Δ completes the proof. \square

The estimates in Theorem 3.1 in turn imply similar bounds on the Hausdorff measure of the m -dimensional manifold M_t , evolving under the flow Φ_t . More precisely, let $\{v_i^x\}_{i=1}^m$ be an orthonormal basis of the tangent space $T_x M$, at the point $x \in M$. Then, writing, as usual, $\mathcal{H}_m(M_t)$ for m -th Hausdorff measure of M_t , we have the following result.

Theorem 3.5 *Let M be a C^2 , m -dimensional manifold, evolving under the flow Φ_t defined in (3). Then under the conditions (A1) – (A4), and for $\alpha = 1 - H + \delta$, $\beta = H - \varepsilon$, such that $(1 - H) < \alpha < 1/2$ and $\delta > \varepsilon$, there exists a constant c_1 , and a random variable $C_{1,T}$, such that*

$$\sup_{t \in [0,T]} \mathcal{H}_m(M_t) \leq c_1 \mathcal{H}_m(M) 2^{C_{1,T} T}.$$

Here $C_{1,T}$ depends on α , β , n , I , and $\{\|B_\gamma^H\|_{\beta,T}, M_\gamma, M_\gamma^{(1)}, M_\gamma^{(2)}\}_{\gamma \in I}$, and satisfies

$$E[C_{1,T}]^\beta \leq C_1 \cdot E[\|B^H\|_{\beta,T}],$$

with the constant C_1 dependent only on α , β , n , $|I|$ and $\{M_\gamma, M_\gamma^{(1)}, M_\gamma^{(2)}\}_{\gamma \in I}$.

Proof: Consider the pushforwards $\{v_{i,t}^x\}_{i=1}^m$ of the tangent vectors $\{v_i^x\}_{i=1}^m$ under the flow Φ_t . Then by using a simple formula for change of variables on a manifold, we have

$$\begin{aligned} \mathcal{H}_m(M_t) &= \int_{M_t} \mathcal{H}_m(dy) \\ &= \int_M \|\alpha^x(t)\| \mathcal{H}_m(dx), \end{aligned}$$

where $\|\alpha^x(t)\| = \sqrt{|\det(\langle v_{i,t}^x, v_{j,t}^x \rangle)|}$. By the Cauchy-Schwartz inequality we know that

$$\langle v_{i,t}^x, v_{j,t}^x \rangle \leq \|v_{i,t}^x\|_2 \|v_{j,t}^x\|_2.$$

Therefore, using Theorem 3.1 and the above expression, we obtain

$$\begin{aligned} \sup_{t \in [0,T]} \|\alpha^x(t)\| &\leq m! \left(\sup_{t \in [0,T]} \|v_{i,t}^x\| \right)^m \\ &\leq c m! 2^{m T C_T}, \end{aligned}$$

which proves the required result. \square

Finally, note that for the case corresponding to $|I| = 1$, the above simplifies to

$$\sup_{t \in [0,T]} \mathcal{H}_m(M_t) \leq c_1 2^{C_1 T} \|B^H\|_{\beta,T}^{1/\beta},$$

for some constants c_1 and C_1 dependent only on the various Lipschitz coefficients of the vector field and its partial derivatives, and this proves Theorem 1.1.

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